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SELF-SIMILAR SOLUTIONS OF THE SECOND-ORDER MODEL OF THE FAR TURBULENT WAKE

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A second-order semi-empirical two-dimensional model of turbulence in the approximation of the far turbulent wake is considered. The sought quantities are the velocity defect, kinetic turbulent energy, energy dissipation, and Reynolds stress. The full group of transformations admitted by this model is found. Self-similar solutions satisfying natural boundary conditions are constructed. The solutions obtained agree with experimental data.

Key words: *far wake, turbulence, self-similarity, second-order models.*

Introduction. Available experimental data [1–3] and numerical calculations [4, 5] show that the flow in the far turbulent wake can be considered to be close to a self-similar flow. Kaptsov and Efremov [6] found self-similar solutions of the k–ε model of turbulence [7]. The objective of the present work was to construct self-similar solutions of the $\overline{u'v'}$–k–ε model [8–10] used to calculate the characteristics of the far turbulent wake behind a plane or an axisymmetric body:

$$u_0\,\frac{\partial u_1}{\partial x}=\frac{1}{y^s}\,\frac{\partial (y^s w)}{\partial y},\qquad u_0\,\frac{\partial k}{\partial x}=\frac{1}{y^s}\,\frac{\partial}{\partial y}\Big(y^s c_\mu\,\frac{k^2}{e}\,\frac{\partial k}{\partial y}\Big)+w\,\frac{\partial u_1}{\partial y}-e,$$

$$u_0\,\frac{\partial e}{\partial x}=\frac{1}{y^s}\,\frac{\partial}{\partial y}\Big(y^s c_\mu\,\frac{k^2}{e}\,\frac{\partial e}{\partial y}\Big)+\frac{e}{k}\Big(c_1 w\,\frac{\partial u_1}{\partial y}-c_2 e\Big), \tag{1}$$

$$u_0\,\frac{\partial w}{\partial x}=\frac{1}{y^s}\,\frac{\partial}{\partial y}\Big(y^s c_s\,\frac{k^2}{e}\frac{\partial w}{\partial y}\Big)-\frac{e}{k}\,c_{f1}w+c_{f2}k\,\frac{\partial u_1}{\partial y}-sc_s\,\frac{k^2}{e}\,\frac{w}{y^2}.$$

Here u_0 is the free-stream velocity, u_1 is the velocity defect, k is the turbulent kinetic energy, e is the kinetic energy dissipation rate, $w=\overline{u'v'}$ is the Reynolds stress, and c_μ, c_s, c_1, c_2, c_{f1}, and c_{f2} are empirical constants; the plane and axisymmetric flows are described by $s=0$ and $s=1$, respectively. In what follows, we assume that the free-stream velocity equals unity.

For system (1), we have to define the admissible differential operators of the point groups of transformations, which will allow us to pass to a system of ordinary differential equations.

1. Self-Similar Solutions. The results of a group analysis of the model performed by a standard scheme [11, 12] are presented below.

Lemma 1. *System* (1) *admits two dilation operators*

$$X_1=x\,\frac{\partial}{\partial x}-u_1\,\frac{\partial}{\partial u_1}-2k\,\frac{\partial}{\partial k}-3e\,\frac{\partial}{\partial e}-2w\,\frac{\partial}{\partial w}; \tag{2}$$

$$Y_1=y\,\frac{\partial}{\partial y}+u_1\,\frac{\partial}{\partial u_1}+2k\,\frac{\partial}{\partial k}+2e\,\frac{\partial}{\partial e}+2w\,\frac{\partial}{\partial w} \tag{3}$$

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and three translation operators in the x, y, and u_1 directions in the case of a plane flow. In the case of an axisymmetric flow, translation in the y direction is not admitted.

In addition, system (1) is invariant with respect to the transformation

$$y \to -y, \qquad w \to -w.$$

Integrating the first equation of system (1) with respect to y from $-\infty$ to $+\infty$, we obtain the conservation law

$$\int_{-\infty}^{\infty} u_1 y^s \, dy = \text{const} \tag{4}$$

with allowance that $y^s w \to 0$ as $y \to \infty$. It should be noted that $w = 0$ outside the turbulent wake.

We write the following linear combination of operators (2) and (3):

$$Z = x\,\partial_x + \alpha y\,\partial_y + (\alpha-1)u_1\,\partial_{u_1} + (2\alpha-2)k\,\partial_k + (2\alpha-3)e\,\partial_e + (2\alpha-2)w\,\partial_w. \tag{5}$$

Then, the solution of system (1) invariant with respect to operator (5) has the form

$$u_1 = x^{(\alpha-1)}U(t), \quad k = x^{(2\alpha-2)}K(t), \quad e = x^{(2\alpha-3)}E(t), \quad w = x^{(2\alpha-2)}W(t), \tag{6}$$

where $t = y/x^\alpha$ is a self-similar variable.

Substituting presentation (6) into (1), we obtain a system of ordinary differential equations

$$\begin{gathered}
W' = -\frac{sW}{t} + \alpha t U' - (\alpha-1)U, \\
K'' = \frac{2(\alpha-1)EK - \alpha t E K' - EWU' + E^2}{c_\mu K^2} + K'\Big(\frac{E'}{E} - \frac{2K'}{K} - \frac{s}{t}\Big), \\
E'' = \frac{(2\alpha-3)E^2K - \alpha t EKE' - c_1E^2WU' + c_2E^3}{c_\mu K^3} + E'\Big(\frac{E'}{E} - \frac{2K'}{K} - \frac{s}{t}\Big), \\
W'' = \frac{2(\alpha-1)EW - \alpha t E W'}{c_s K^2} + \frac{c_{f1}E^2W - c_{f2}EK^2U'}{c_s K^3} + W'\Big(\frac{E'}{E} - \frac{2K'}{K} - \frac{s}{t}\Big) + \frac{sW}{t^2}.
\end{gathered} \tag{7}$$

By virtue of the conservation law (4) and presentation (6) for the function u_1, we obtain $\alpha = 1/2$ for a plane flow and $\alpha = 1/3$ for an axisymmetric flow. Hence, the self-similar variable t equals $y/x^{1/2}$ or $y/x^{1/3}$, depending on the type of symmetry of the problem. Note that this self-similarity is validated by experimental data [3].

Using the first equation of system (7), we find the first integral:

— for a plane flow,

$$W + \alpha t U = b_1, \qquad b_1 \in \mathbb{R}; \tag{8}$$

— for an axisymmetric flow,

$$Wt + \alpha t^2 U = b_2, \qquad b_2 \in \mathbb{R}. \tag{9}$$

System (7) has to satisfy the conditions

$$W(0) = 0, \qquad U'(0) = K'(0) = E'(0) = 0; \tag{10}$$

$$U(1) = K(1) = E(1) = W(1) = 0. \tag{11}$$

Condition (10) takes into account flow symmetry with respect to the OY axis. The boundary conditions (11) imply that all functions take zero values outside the turbulent wake.

Note that system (7) admits the dilation operator

$$T = t\,\frac{\partial}{\partial t} + U\,\frac{\partial}{\partial U} + 2K\,\frac{\partial}{\partial K} + 2E\,\frac{\partial}{\partial E} + 2W\,\frac{\partial}{\partial W},$$

therefore, conditions (11) can be replaced by similar conditions at any arbitrary point, for instance, at the point $a = 0.4$.

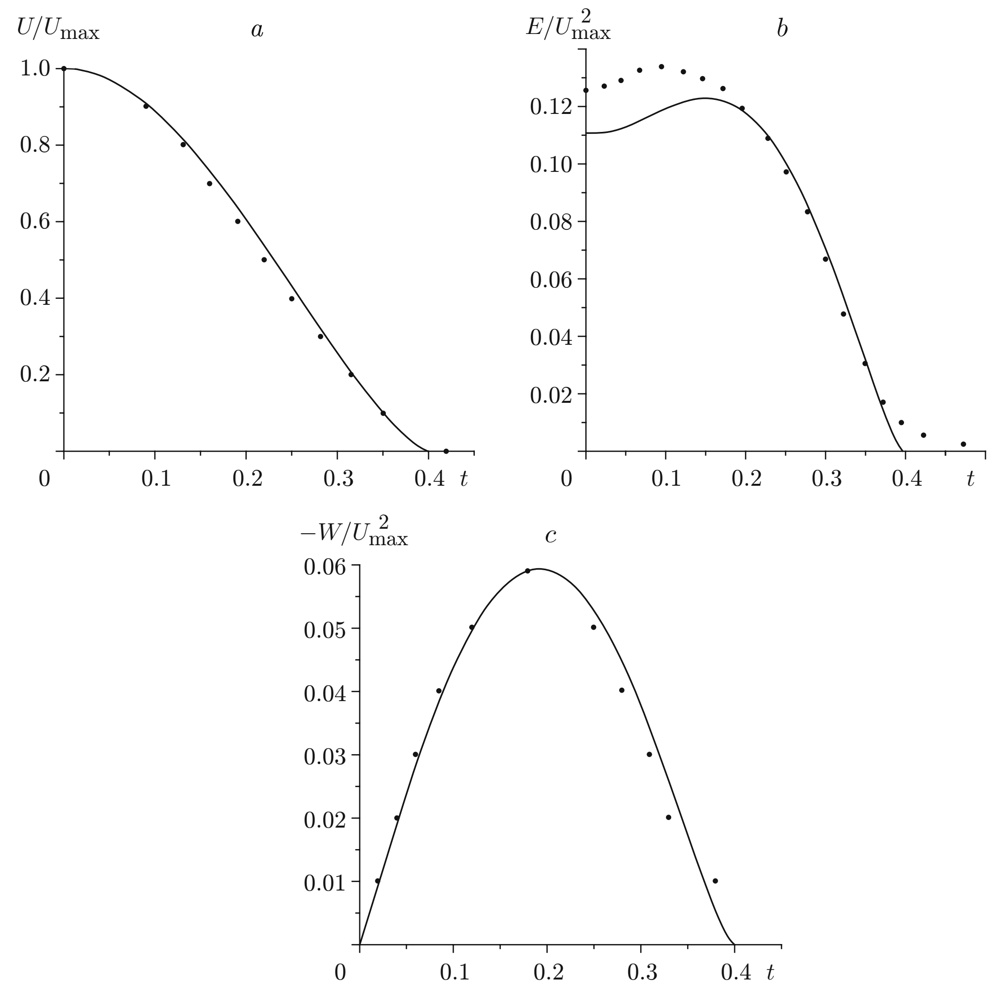

Fig. 1. Functions $U/U_{\max}$ (a), $E/U_{\max}^2$ (b), and $-W/U_{\max}^2$ (c) for a plane flow: the curves and points refer to the calculated results and the experimental data of [3], respectively.

Using the boundary conditions (10) and (11), we can easily demonstrate that the constants b_1 and b_2 equal zero in integrals (8) and (9).

The first integral allows us to eliminate the function W from system (7) and obtain a system of three ordinary differential equations of the second order

$$U'' = \frac{(\alpha-2)EU - \alpha tEU'}{c_s K^2} + \frac{c_{f1}E^2U}{c_s K^3} + \frac{c_{f2}EU'}{c_s\alpha tK} - \frac{(s+2)U'}{t} + \Big(\frac{E'}{E} - \frac{2K'}{K}\Big)\Big(U' + \frac{U}{t}\Big),$$

$$K'' = \frac{2(\alpha-1)EK + \alpha tE(UU' - K') + E^2}{c_\mu K^2} + K'\Big(\frac{E'}{E} - \frac{2K'}{K} - \frac{s}{t}\Big), \tag{12}$$

$$E'' = \frac{(2\alpha-3)E^2K - \alpha tEKE' + c_1\alpha tE^2UU' + c_2E^3}{c_\mu K^3} + E'\Big(\frac{E'}{E} - \frac{2K'}{K} - \frac{s}{t}\Big).$$

The remaining boundary conditions have the form

$$U'(0) = K'(0) = E'(0) = U(1) = K(1) = E(1) = 0. \tag{13}$$

2. Calculation Results. System (12) satisfying conditions (13) was solved numerically. The following empirical constants were used: $c_\mu = 0.09$, $c_1 = 1.44$, $c_2 = 1.92$, $c_s = 0.09$, $c_{f1} = 2.8$, and $c_{f2} = c_\mu c_{f_1}$. Additional difficulties are caused by the fact that the coefficients of system (12) have singularities at $t = 0$ and 1. The problem

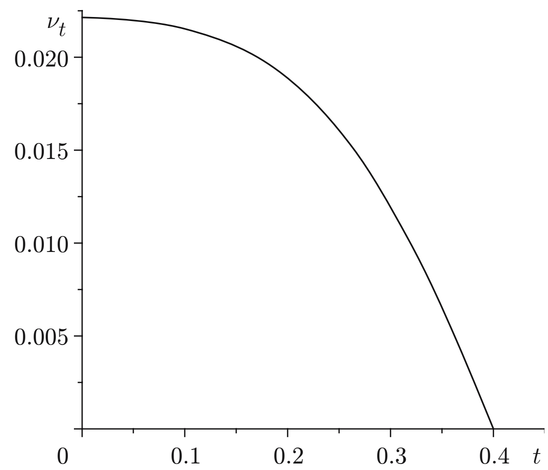

Fig. 2. Profile of eddy viscosity for a plane flow.

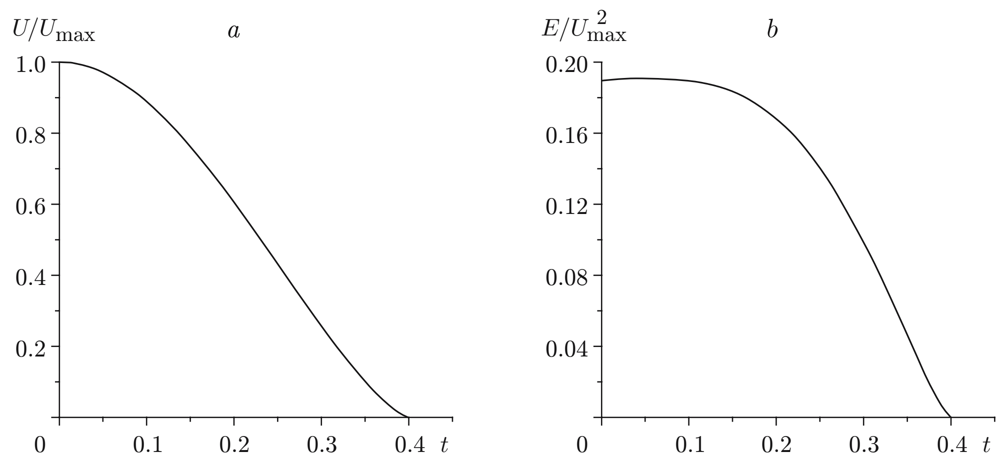

Fig. 3. Functions $U/U_{\max}$ (a) and $E/U^2_{\max}$ (b) for an axisymmetric flow.

was solved by a modified shooting method and asymptotical expansion of the solution in the vicinity of the singular point [13].

The functions $U/U_{\max}$, $E/U^2_{\max}$, and $-W/U^2_{\max}$ versus a self-similar variable for a plane flow are plotted in Fig. 1. Figure 2 shows the profile of eddy viscosity $\nu_t = c_\mu K^2/E$. The results of the problem solution for an axisymmetric flow are illustrated in Fig. 3. The value $t = 0.4$ is taken in the boundary conditions (13) instead of $t = 1$. For a plane flow, the distributions of velocity and turbulence energy with respect to the parameter t are in reasonable agreement with the experimental data [3].

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